Can mathematical modelling help geothermal resources exploitation?

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Outline

The Macgeo research team

Title: "**Ma**thematical modelling for government control of public **C**oncession (license) for exploitation of **GEO**thermal resources"

Total funds: ≈800,000 € *source*: Tuscany Government

Scientific coordinator: F. R.

Research group in the Math. Dept.: I. Borsi, M. Cerminara, M. Ceseri, A. Farina, A. Fasano, L. Fusi, L. Meacci, M. Primicerio, A. Speranza

National institutions involved: Dept. of Math. Ulisse Dini, I2T3, Dept. Earth Sciences, Dept. Informatics Systems, Media Integration and Communication Center, Dept. of Chem. Engineering, Mines and Environmental Technologies, Institute of Bio-Meteorology, National Interuniversitary Consortium for the Engineering of Geo-resources

The MACGEO project

It has a multipurpose and ambitious plan

Main goal: to develop simulation models integrated with a Geo Data Base, to furnish Regione Toscana with a useful tool to evaluate the exploitation of geothermal fields (Larderello and Amiata) and their impact on the environment.

The MACGEO project

Interesting results:

- New methods to write input files for the code Tough2,
- new calibration algorithms developed for Tough2 (alternative to the iTough2 modules),
- a new web-based Geo DB specifically developed for the Tuscany geothermal fields,
- new thermodynamic module for iTough2,
- **specific modeling activity to better understand the reservoir processes**.

The MACGEO official web-page

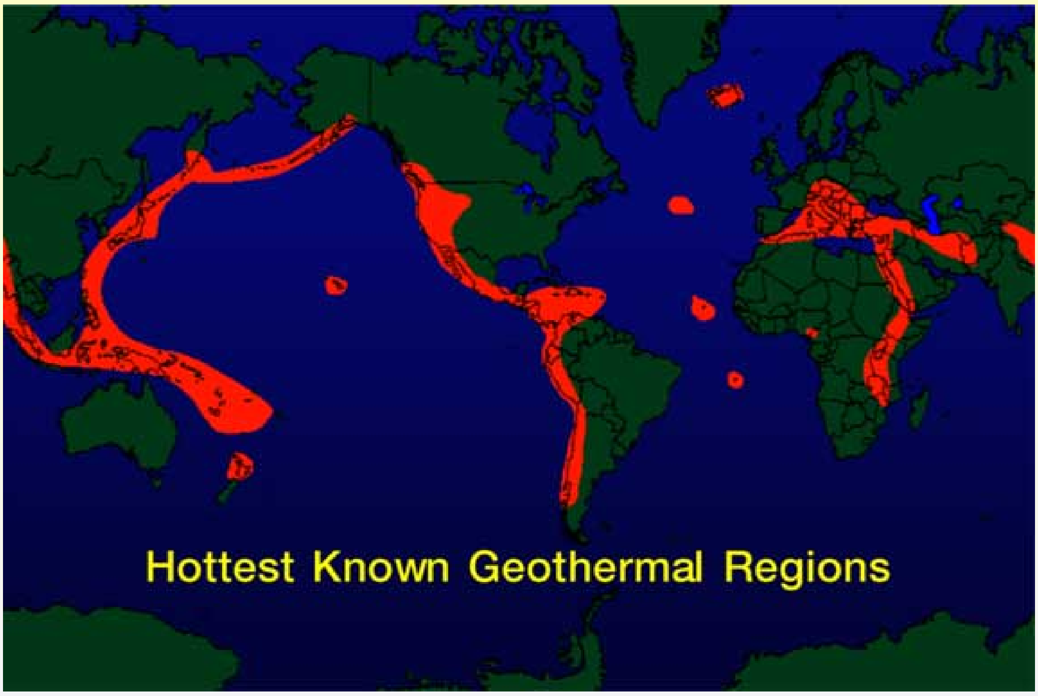
GTE and the Earth heat flux

Total heat power of the earth is evaluated to be 42 TW (4.2×10^{13} J/s) but the heat flux (coming mostly from decay of natural radioactive elements) is not the same everywhere!

The heat flux is higher where the crust is thinner! (that is along tectonic plates boundaries and subduction zones).

Standard geothermal gradient $\nabla T \approx 2 \div 3°\text{C}/100$ m; geothermal areas $\nabla T \geq 7°\text{C}/100$ m.

GTE and the Earth heat flux



History of GTE exploitation

GTE **at low enthalpy** (thermal springs) has been known for centuries and used by animals and humans.

GTE **at high enthalpy** is relatively recent (4 July 1904):

Prince Piero Ginori Conti (son in law of Earl Florestano de Larderel) tested the first geothermal power generator at the Larderello dry steam field in Italy

Actual development of GTE

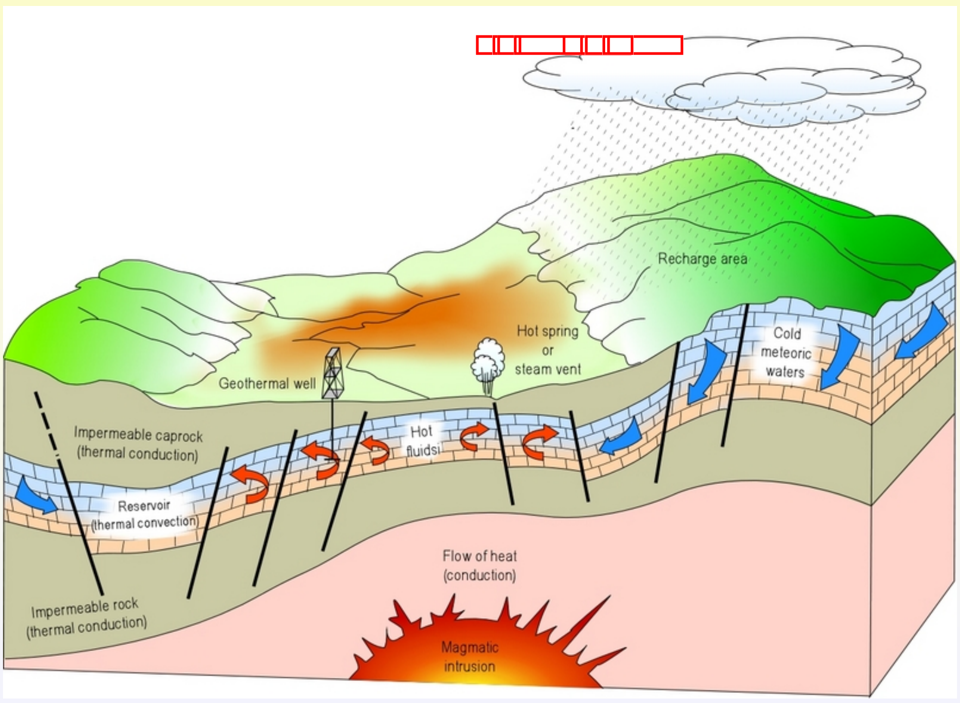
1911: the world's first geothermal power plant (250 KW) was built in the Devil's Valley of Larderello.

This remained the world's only industrial producer of GTE until 1958 when New Zealand built a plant of its own in Wirakei

Actually there geothermal power plants in 24 countries. Current worldwide installed capacity is about 11 gigawatts (GW), with the largest capacity in the United States (3 GW) Philippines (1.9 GW), Indonesia (1.1 GW), Mexico (0.9 GW) and Italy (0.85 GW).

Classical model of a geothermal basin

It works like a pressure pot!



Mathematical and physical complexity

Full-field 3-D modelling of a geothermal reservoir is possible but extremely complex!

Only advanced numerical codes can manage the highly nonlinear equations involved.

Indeed a geothermal fluid is a **multicomponent** (for example H_2O plus partially dissolved CO_2 and $NaCl$) and **multiphase** (liquid, vapour, solid) system.

The three phases co-exist in the porous matrix (the reservoir deep rocks) in local thermodynamic equilibrium (with respect to the long time scale of evolution of the reservoir).

Thermodynamic parameters are close to criticality

This is why

simpler mathematical models are very useful!

These models allow more mathematical analysis and suggest interesting conclusions.

Classical (well-known) Darcy's law for porous media: **the flow velocity is linearly related to the pressure gradient**.

First "simple" problem:

fluid motion at borehole scale: Darcian or not?

Ref:

Borsi I., Fusi L., Rosso F., Speranza A., *Isothermal two-phase flow of a vapor-liquid system with non-negligible inertial effects* Int. J. Eng. Sci. in press

Borsi I., Fusi L., Rosso F., Speranza A., *A well deliverability model for multi-phase non-Darcian flow in geothermal reservoirs*, Computer and Geosciences, in press.

Non-Darcian flow near the borehole

Two phase flow : geothermal fluid is **mono**-component (say "water"), vapour and liquid phases co-exist in a undeformable porous matrix.

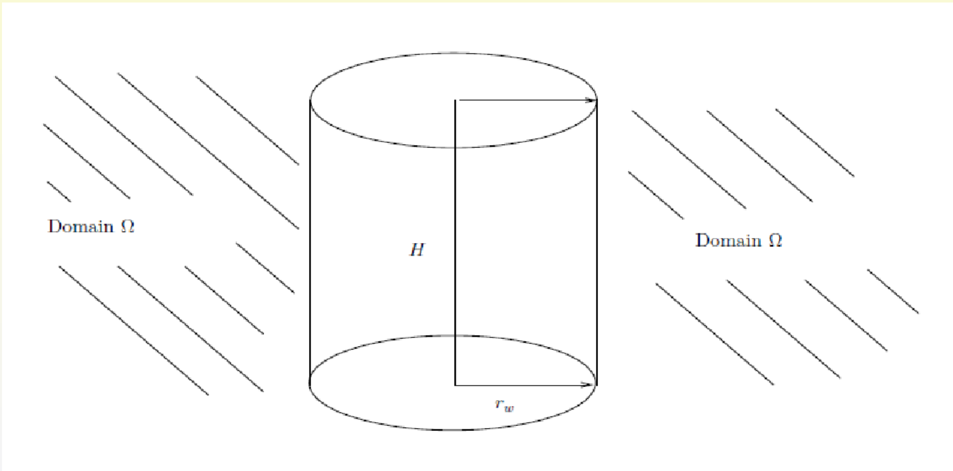
Main practical application: modelling the region near the borehole in a geothermal reservoir.

Main assumptions

- Isothermal
- Stationary
- Axisymmetric cylindrical geometry
- Nonlinear flow (Forchheimer instead of Darcy)
- Non negligible Capillary Pressure

Local borehole geometry

External cylindrical domain: H is the height of productive zone, generally **three order of magnitude** smaller than the height of the well! ⇒**plane, axisymmetric motion**



Mass balance and fluid flow

Axial symmetry plus continuity equation

$$\rho^l u^l + \rho^g u^g = -\left(\frac{w}{2\pi H}\right)\frac{1}{r}, \quad r \in [r_w, \infty).$$

w = amount of fluid extracted per unit time at $r = r_w$, u = superficial velocity, ϱ = density, g, l = gas, liquid

Nonlinear Forchheimer's model (with gravity neglected)

$$\frac{\mu^\alpha}{k k_r^\alpha(S^l)} u^\alpha + \beta \rho^\alpha |u^\alpha| u^\alpha = -\frac{dP^\alpha}{dr}, \qquad r \in [r_w, \infty).$$

k = absolute permeability, k_r^α = relative permeability of phase α, μ = viscosity, S^l = liquid saturation, β = Forchheimer parameter

Equations of state

Ideal Gas

$$\rho^g = \frac{M_w P^g}{RT} =: \frac{\gamma P^g}{T},$$

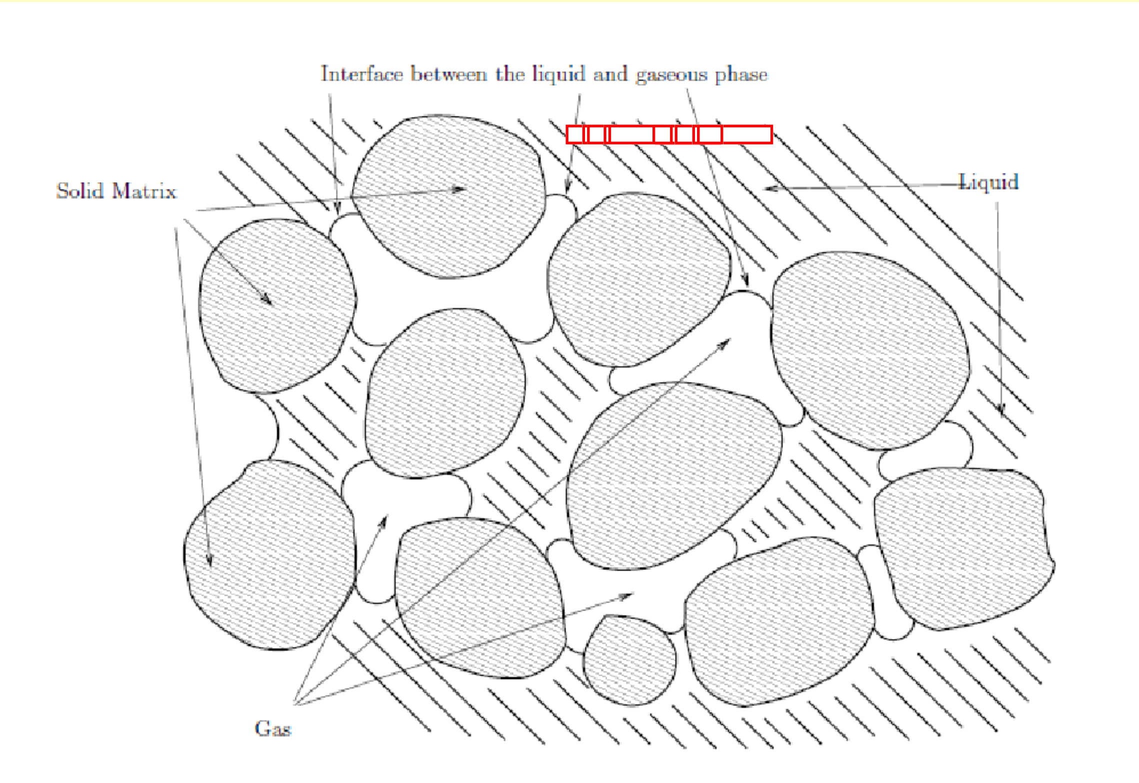
Slightly compressible fluid (linearized exponential law)

$$\rho^l = \rho_o \left[1 + \lambda(P^l - P_o)\right],$$

Only two phases

$$S^l + S^g = 1$$

Capillary pressure



Capillary pressure

d = average pore diameter, σ = surface tension, θ = contact angle

$$P_c = \frac{2\sigma}{d}\cos\theta = \underbrace{P_{\mathrm{nw}}}_{\text{non wetting phase}} - \underbrace{P_{\mathrm{w}}}_{\text{wetting phase}}$$

At thermodynamic equilibrium: **when $P_c = 0$**

$$P^g = P^l = P_{sat}(T)$$

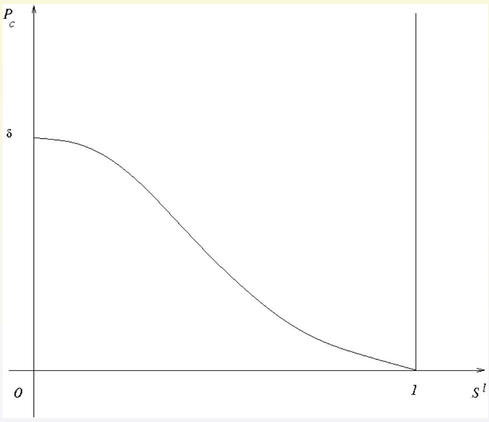
where $P_{sat}(T)$ is known (Clausius-Clapeyron relation). **When $P_c \neq 0$**: vapour pressure lowering effect (Kelvin's equation)

$$P^g = P_{sat}\exp\left(-\frac{P_c\gamma}{\rho^l T}\right) \leqslant P_{sat},$$

Capillary pressure: typical behaviour

Within rocks liquid is the wetting phase: thus at thermodynamic equilibrium $P_c = P^g - P^l$

At macroscopic level capillary pressure has to be constitutively assigned as a function of liquid saturation S^l:



- $P_c > 0$ ($\Leftrightarrow \theta \in (0, \pi/2)$)
- $P_c' \leqslant 0$
- $P_c(1) = 0, \quad P_c(0) = \delta$

The mathematical model

Dimensionless equations (pressure is rescaled with P_{sat})

$$\begin{cases} \rho^g = \psi P^g = \psi \exp\left(-\dfrac{P_c\psi}{\rho^l}\right), \\ \rho^l = 1 + \eta(P^l - 1) = 1 + \eta(P^g - P^c - 1), \\ \rho^g u^g + \rho^l u^l = -\dfrac{1}{r}, \qquad r \in [1, \infty) \\ (\theta^\alpha + \varepsilon \rho^\alpha |u^\alpha|) u^\alpha = -\dfrac{dP^\alpha}{dr}, \qquad r \in [1, \infty), \qquad \alpha = l,\ g. \end{cases}$$

ψ, η, ε are dimensionless, positive coefficients coming out from scaling.

The mathematical model (non-dimensional)

Eliminate P_g

$$\rho^l = 1 + \eta\left[\exp\left(-\frac{P_c\psi}{\rho^l}\right) - P_c - 1\right].$$

Write velocity as a function of ∇P (Forchheimer)

$$\rho^\alpha u^\alpha = \operatorname{sgn}\left(\frac{dP^\alpha}{dr}\right)\left[\frac{\theta^\alpha - \sqrt{(\theta^\alpha)^2 + 4\left|\dfrac{dP^\alpha}{dr}\right| \varepsilon\rho^\alpha}}{2\varepsilon}\right], \qquad \alpha = l,\ g.$$

where θ^α are **known functions of S^α** (via $k_r^\alpha(S^l)$) and P^α, ρ^α are **known functions of S^l** (via $P_c = P_c(S^l)$).

Differential equation for liquid saturation $S^l(r)$

Substitute into the mass balance

$$\sum_{\alpha=l,g} \text{sgn}\left(\frac{dP^\alpha}{dr}\right)\left[\theta^\alpha - \sqrt{(\theta^\alpha)^2 + 4\left|\frac{dP^\alpha}{dr}\right|\varepsilon\rho^\alpha}\right] = -\frac{2\varepsilon}{r}.$$

Use $\frac{dP^\alpha}{dr} = \frac{dP^\alpha}{dS^l}\frac{dS^l}{dr}$ to get the following **highly nonlinear ODE**:

$$F\left(\frac{dS^l}{dr}, S^l, r\right) = 0$$

to be solved with initial datum $S^l(1) \in [0,1]$ ($r = 1$ is the wall of the drilled borehole). **To find $S^l(r)$ means to identify the liquid-vapour co-existence domain!** This equation **is too complex to treat even numerically! We try asymptotics.**

Approximated problem

Look for a solution

$$S^l = S^l_o + \varepsilon S^l_1 + \frac{\varepsilon^2}{2!}S^l_2 + \ldots.. = \sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}S^l_i.$$

$$\varepsilon = \frac{\beta w^2}{(2\pi H)^2\rho_{sat}P_{sat}r_w}$$

and expand nonlinear terms around $\varepsilon = 0$.

Reformulate the problem neglecting the nonlinear terms

$$P^\alpha(S^l) = P^\alpha_o + P^\alpha_1\varepsilon S^l_1, \qquad u^\alpha(S^l) = u^\alpha_o + u^\alpha_1\varepsilon S^l_1,$$

$$\rho^\alpha(S^l) = \rho^\alpha_o + \rho^\alpha_1\varepsilon S^l_1, \qquad \theta^\alpha(S^l) = \theta^\alpha_o + \theta^\alpha_1\varepsilon S^l_1,$$

Differential equations - Zero Order Approximation

Zero order approximation (Darcian flow)

$$\rho_o^g u_o^g + \rho_o^l u_o^l = -\frac{1}{r}, \qquad \theta_o^\alpha u_o^\alpha = -\frac{dP_o^\alpha}{dr}, \qquad \alpha = l,\ g,$$

Eliminating velocities yields a **nonlinear** (all coefficients

depend on S_o^l) first order ODE

$$\left(\frac{\rho_o^g}{\theta_o^g}\frac{dP_o^g}{dS_o^l} + \frac{\rho_o^l}{\theta_o^l}\frac{dP_o^l}{dS_o^l}\right)\frac{dS_o^l}{dr} = \frac{1}{r}. \qquad \Longrightarrow \qquad S_o^l(r)$$

which allows to compute (only numerically) S_o^l for a given initial datum $S_o^l(1) \in [0,1]$.

Differential equations - First Order Approximation

Simplified Forchheimer's flow

$$\rho_o^g u_1^g + \rho_1^g u_0^g + \rho_o^l u_1^l + \rho_1^l u_o^l = 0$$

$$\theta_o^\alpha u_1^\alpha S_1^l + \theta_1^\alpha u_o^\alpha S_1^l + \rho_o^\alpha |u_o^\alpha| u_o^\alpha = -\frac{d}{dr}(P_1^\alpha S_1^l), \quad \alpha = l,\ g,$$

Eliminating velocities as before yields a **linear differential**

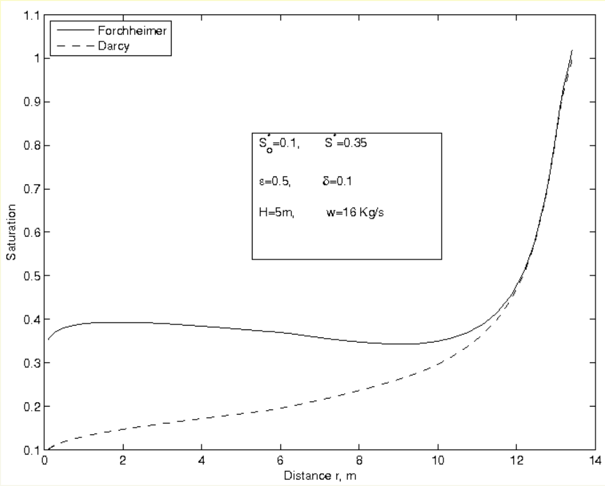
equation

$$\frac{dS_1^l}{dr} = \mathcal{F}_1(S_o^l(r))S_1^l + \mathcal{F}_2(S_o^l(r)), \qquad \Longrightarrow \qquad S_1^l(r)$$

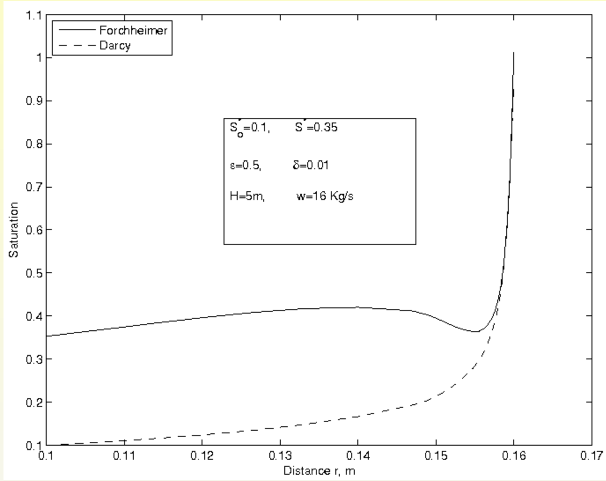
Functions $\mathcal{F}_1, \mathcal{F}_2$ are rather complex: only numerical integration is possible.

Numerical simulations - Plot $S_o^l(r)$ and $S_o^l(r) + \varepsilon S_1^l(r)$

Influence of capillary pressure ($\delta = P_c(0) > 0$)



Phase coexistence up to $r \approx 13$ m from $r = r_w$ when $\delta = 0.1$

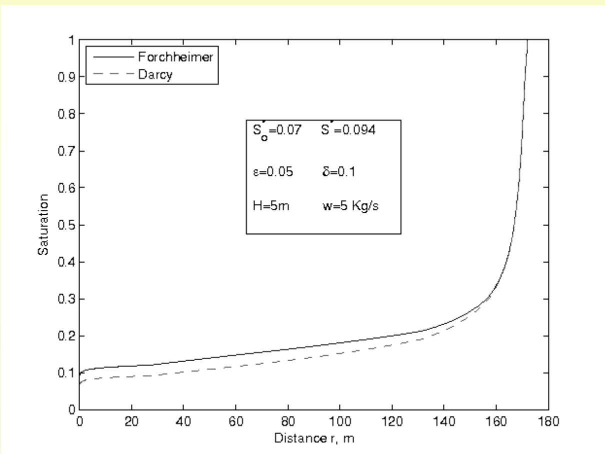


Phase coexistence up to $r \approx 0.16$ m from $r = r_w$ when $\delta = 0.01$

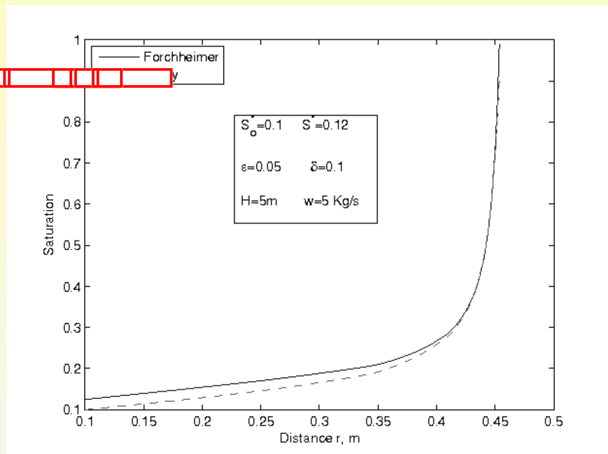
Forchheimer very different from Darcy!

Numerical simulations - Plot $S_o^l(r)$ and $S_o^l(r) + \varepsilon S_1^l(r)$

Influence of the initial datum



$S_o^l(0) + \varepsilon S_1^l(0) = 0.117.$
Co-existence up to ≈ 170 m from $r = r_w$



$S_o^l(0) + \varepsilon S_1^l(0) = 0.16.$
Co-existence up to ≈ 0.45 m from $r = r_w$

Dramatic change of the phase co-existence domain

Implications

1. The two models (Darcy and Forchheimer) differs significantly in the domain of co-existence ($0 < S^l < 1$). **Neglecting inertial effects can be misleading!**
2. A decrement of $\delta = P_c(0) > 0$ produces a decrease in the size of the domain of co-existence. As $\delta \to 0$ this domain reduces to a point.
3. The width of the co-existence domain is extremely sensitive to the initial datum. Specifically, as $S^l_o \to 0$, the coexistence domain becomes $[r_w, \infty)$

Conclusions are interesting but we can do something better! Let's try another, more advantageous, approach based on "parent density".

Second "simple" problem: same question as before but

fully non linear analysis by means of the overall fluid density (mono-component case)!

Weak formulation for a two-phase nonlinear flow of a compressible fluid in an undeformable porous medium (Borsi I., Ceseri M., Fusi L., Rosso F., Speranza A., 2011)

Isothermal two-phase flow of a vapor-liquid system with non-negligible inertial effects Borsi I., Fusi L., Rosso F., Speranza A., 2011)

The "parent density" approach

Now we definitely abandon Darcy's law in favor of Forchheimer and forget about asymptotics! The fluid is still **mono-component** and all physical assumptions are the same as before.

First step: define the **parent density**

$$\rho = \rho^l S^l + \rho^g S^g = \rho^l S^l + \rho^g (1 - S^l). \qquad (1)$$

Clearly $\rho = \rho(S^l)$ non-linearly since ρ^l, ρ^g depend on S^l.

Idea: work out analysis using ρ instead of S^l as the main thermodynamic variable of the problem!

The "parent density" approach

Under **reasonable assumptions** it turns out

$$\frac{d\rho}{dS^l} = \left[\rho^l - \rho^g + \frac{\partial \rho^l}{\partial S^l} S^l + \frac{\partial \rho^g}{\partial S^l}(1 - S^l)\right] > 0$$

$\rho(S^l)$ is therefore **invertible** and we can write $S^l = S^l(\rho)$: the domain of the function $S^l(\rho)$ is the codomain of the function $\rho(S^l)$. Therefore

$$\rho(0) \leqslant \rho \leqslant \rho(1). \qquad (2)$$

The "parent density" approach

Being (we use scaled dimensionless variables)

$$\rho(0) = \rho^g(0) = \psi \exp\left(-\frac{P_c(0)\psi}{\rho^l(0)}\right) =: \rho_{min}, \tag{3}$$

$$\rho(1) = 1 \tag{4}$$

we have

$$0 < \rho_{min} \leqslant \rho \leqslant 1, \tag{5}$$

where $\rho^l(0)$ is evaluated through the implicit relation

$$\rho^l = 1 + \eta \left[\underbrace{\overbrace{\exp\left(-\frac{P_c\psi}{\rho^l}\right)}^{\text{Kelvin eq.}} - P_c}_{\text{liquid pressure}} - 1 \right].$$

The "parent density" approach

Use the parent density ρ to determine the phase of the fluid:

$$\begin{cases} \rho \geqslant 1 & \textbf{only liquid} \\ \rho_{min} < \rho < 1 & \textbf{liquid and gas} \\ \rho \leqslant \rho_{min} & \textbf{only gas} \end{cases} \tag{6}$$

Notice that when $P_c \equiv 0$ the co-existence phase disappears, since $\rho^l(0) = 1$ and $\rho^g(0) = 1$.

The "parent density" approach

For simplicity we assume that the spatial domain is the set $\Omega = [0, L]$ though minor changes allows to consider also the axisymmetric cylindrical case.

Second step: solve Forchheimer's eq. in the form

$$\rho^\alpha u^\alpha = \operatorname{sgn}\left(\frac{\partial P^\alpha}{\partial x}\right)\left[\frac{\theta^\alpha - \sqrt{(\theta^\alpha)^2 + 4\left|\frac{\partial P^\alpha}{\partial x}\right|\varepsilon\rho^\alpha}}{2\varepsilon}\right], \qquad (7)$$

for $\alpha = l,\ g$, where θ^α is a **known** dimensionless function of S^l, ε is the dimensionless Forchheimer parameter.

Parabolic equation for the "parent density"

Third step: express the **total mass flux** as a unique function.:

$$W\left(\rho, \frac{\partial \rho}{\partial x}\right) = \begin{cases} 0, & \rho \leqslant 0, \\ W_1(\rho, \rho_x), & 0 < \rho \leqslant \rho_{min}, \\ W_2(\rho, \rho_x), & \rho_{min} < \rho < 1, \\ W_3(\rho, \rho_x), & \rho \geqslant 1, \end{cases} \qquad (8)$$

where W_1, W_2, W_3 are complicated **but known** functions.

Forth step: rewrite mass conservation using (7) and W to get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[W\left(\rho, \frac{\partial \rho}{\partial x}\right)\right] = 0. \qquad (9)$$

Notice that $W(\rho, 0) = 0$.

Free boundary conditions

The interfaces where the fluid switches phase are

$$\begin{cases} \rho = \rho_{min} & \textbf{Gas} \quad \longleftrightarrow \quad \textbf{Liquid} + \textbf{gas} \\ \rho = 1 & \textbf{Liquid} + \textbf{gas} \quad \longleftrightarrow \quad \textbf{Liquid} \end{cases} \tag{10}$$

We impose the **continuity of the overall mass flux across the free boundaries** $\rho = \rho_{min}$ **and** $\rho = 1$.

This yields

Free boundary conditions

(free boundary condition on $\rho = \rho_{min}$)

$$\operatorname{sgn}\left(\frac{\partial \rho^-}{\partial x}\right)\left(\hat{\theta}^g - \sqrt{\left(\hat{\theta}^g\right)^2 + \frac{4}{\psi}\left|\frac{\partial \rho^-}{\partial x}\right| \varepsilon \rho_{min}}\right)$$

$$= \operatorname{sgn}\left(\frac{\partial \rho^+}{\partial x}\right)\left(\hat{\theta}^g - \sqrt{\left(\hat{\theta}^g\right)^2 + 4 J^g(\rho_{min})\left|\frac{\partial \rho^+}{\partial x}\right| \varepsilon \rho_{min}}\right) \tag{11}$$

(where - and + here stands for the limits from the gaseous and gaseous+liquid phase respectively), and

$$0 < J^g(\rho_{min}) = \left[\frac{\dfrac{1}{\psi}\dfrac{\partial \rho^g}{\partial S^l}}{\rho^l - \rho^g + \dfrac{d\rho^g}{dS^l}}\right]\Bigg|_{\rho=\rho_{min}} < \infty$$

Free boundary conditions

(free boundary condition on $\rho = 1$)

$$\operatorname{sgn}\left(\frac{\partial \rho^-}{\partial x}\right)\bigg|_{\rho=1}\left(\hat{\theta}^l - \sqrt{\left(\hat{\theta}^l\right)^2 + \frac{4}{\eta}\left|\frac{\partial \rho^-}{\partial x}\right|\bigg|_{\rho=1}\varepsilon}\right)$$

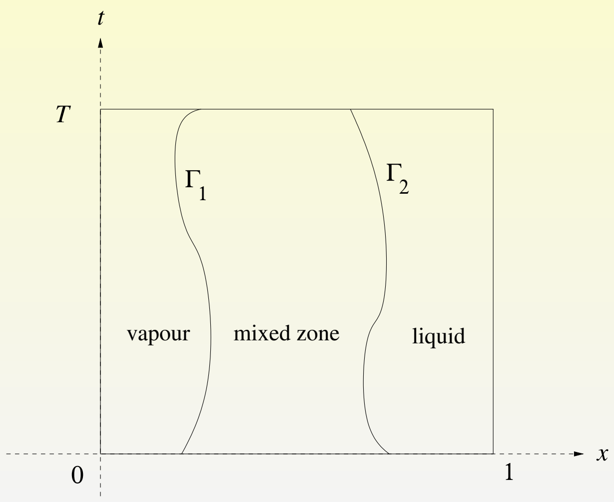
$$= \operatorname{sgn}\left(\frac{\partial \rho^+}{\partial x}\right)\bigg|_{\rho=1}\left[\hat{\theta}^l - \sqrt{\left(\hat{\theta}^l\right)^2 + 4J^l(1)\left|\frac{\partial \rho^+}{\partial x}\right|\bigg|_{\rho=1}\varepsilon}\right] \tag{12}$$

(where - and + here stands for the limits from the liquid and gaseous+liquid phase respectively), and

$$0 < J^l(1) = \left[\frac{\frac{1}{\eta}\frac{\partial \rho^l}{\partial S^l}}{\rho^l - \rho^g + \frac{d\rho^l}{dS^l}}\right]\Bigg|_{\rho=1} < \infty.$$

Free boundary problem

Conditions (11) and (12) close the problem, which is a quasilinear parabolic free boundary problem. The free boundary conditions are $\rho = \rho_{min}$ and (11), $\rho = 1$ and (12).



The number of free boundaries is not a priori known.
Example: only two free boundaries, Γ_1 (where $\rho = \rho_{min}$) and Γ_2 (where $\rho = 1$)

Weak formulation

Multiply equation (9) by a smooth test function $\phi \in C^\infty(\bar{\Omega}_T)$ such that

$$\phi(0,t) = \phi(L,t) = 0, \qquad \forall t \in [0,T], \tag{13}$$

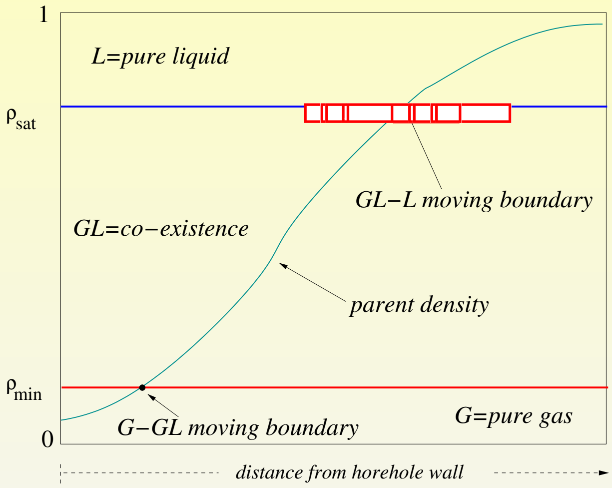
$$\phi(x,T) = 0, \qquad \forall x \in [0,1]. \tag{14}$$

and integrate over Ω_T to get

$$\int_{\Omega_T} \left[\frac{\partial \phi}{\partial t} \rho + \frac{\partial \phi}{\partial x} W \right] dxdt = - \int_0^L (\phi\rho)(x,0) dx. \tag{15}$$

Because of continuity of W across the interfaces the same formulation is obtained if even a countable set of free boundaries may develop.

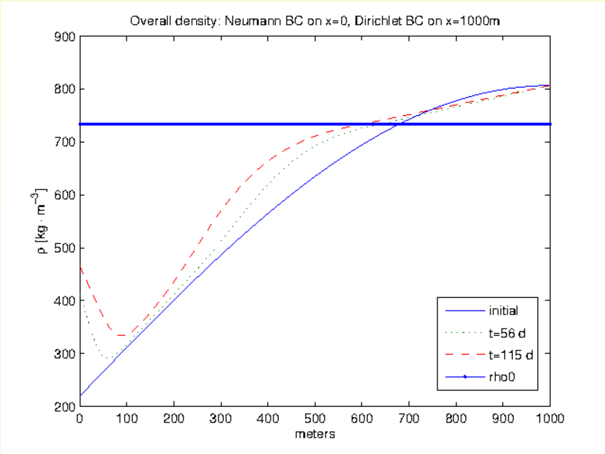
Pictorial view of the free boundaries



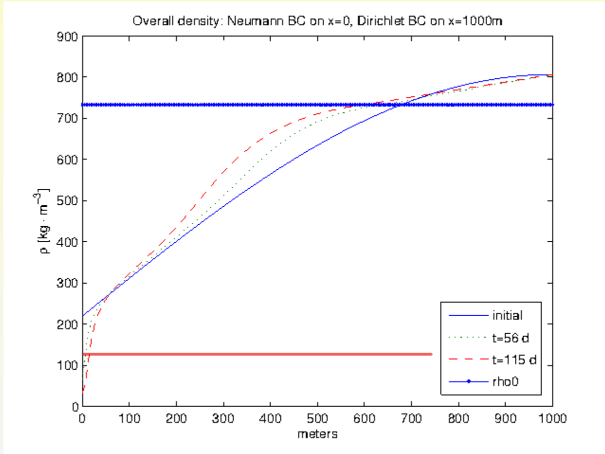
If $\rho(r)$ is not monotonic more than two free boundaries may appear!

Numerical simulations (with real field data)

Characteristic time ≈ 193 d, $\rho_{sat} \approx 733\ Kg \cdot m^3$.
$\rho = 806\ Kg \cdot m^3$ on $x = L$, $\rho_{\min} \approx 133\ Kg \cdot m^3$,
$W_u = 1\ Kg \cdot m^{-2} \cdot s^{-1}$ (flux unit)

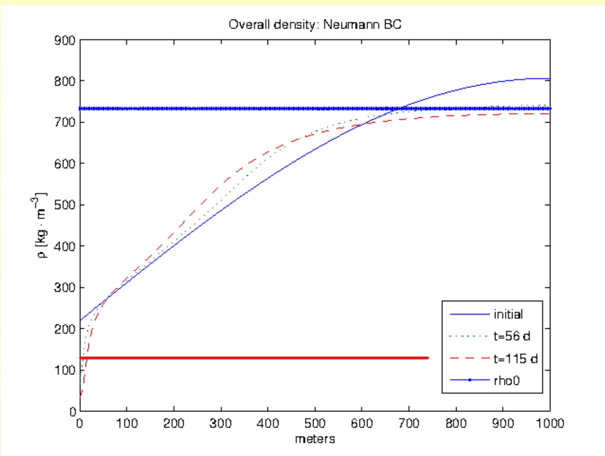


$W(x = 0) = 5 \cdot 10^{-2} W_u$ (injection of fluid). A pure gas phase is never present

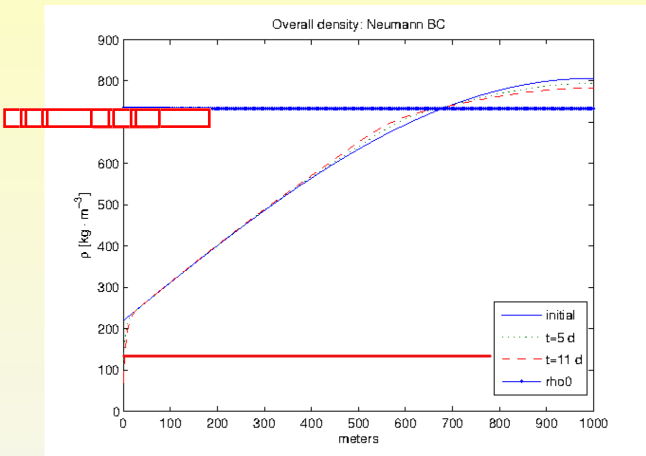


$W(x = 0) = -5 \cdot 10^{-2} W_u$ (extraction of the fluid). A pure gas phase appears for $t > 100$ d

Numerical simulations



$W(x = 0) = -5 \cdot 10^{-2}\ W_u$,
$W(x = L) = 0\ W_u$.
A pure gas phase appears for $t > 100$ d



$W(x = 0) = -10^{-1}\ W_u$,
$W(x = L) = 0\ W_u$.
A pure gas phase appears for $t > 7$ d

Previous problem fully extended to 3D and multi-component flows:

see *Isothermal two phase flow of a multi-component fluid through a porous media when capillary phenomena are non-negligible and the flow is governed by Forchheimer's law* (Fusi L., Farina A., Rosso F. 2011)

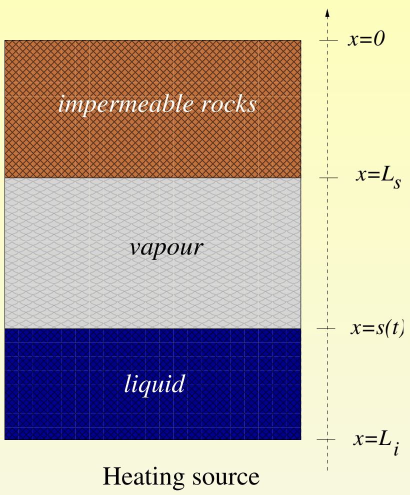
Another problem: large-scale time behaviour of a reservoir

(e.g. Larderello is vapour-dominated. Mt. Amiata is water-dominated, Monteverdi Marittimo has a mixed situation...**Why?**)

A simple one-dimensional model
(L. Meacci 2008, MSc-thesis)

Main hypotheses

1. Geothermal fluid is **mono-component** (say water) (l =liquid or v =vapour)
2. **No capillary effects**: S^l can be only 0 or 1 (different phases sharply separated by an interface at $z = s(t)$)
3. **Constant porosity**
4. **Constant permeability**
5. **1-D geometry**
6. **use Darcy's law**
7. **reservoir is isolated**



The 1-D domain is $[L_i, L_s]$:

$S^g = 1, S^l = 0$ for $x > s(t)$.

$S^g = 0, S^l = 1$ for for $x < s(t)$.

Temperature changes linearly:

$$T(x) = T_i - \frac{T_i - T_s}{L_i - L_s}(L_i - x).$$

At the upper boundary $x = L_s$ pressure P_s is constant,

at the lower boundary $x = L_i$ flux is null ($v^l = 0$)

Model eqs

for the **liquid domain** $L_i < x < s(t)$

Recall: ρ_l, $\phi=$ constant and $v_l|_{x=L_i} = 0$. Thus

- **mass balance**:

$$\frac{\partial}{\partial t}(\phi\rho_l) + \frac{\partial}{\partial x}(\phi\rho_l v_l) = 0, \quad \Rightarrow \quad v_l = 0, \quad x \in (L_i, s(t))$$

- **Darcy's law**:

$$v_l = -\frac{K}{\phi\mu_l}\left(\frac{\partial P_l}{\partial x} + \rho_l g\right) = 0 \;\Rightarrow\; \frac{\partial P_l}{\partial x} = -\rho_l g,$$

- **solution** (with $P_l(s(t))$ to be specified):

$$P_l(x) = \underbrace{P_l(s(t))}_{\text{ph. change press}} + \underbrace{\rho_l g(s(t) - x)}_{\text{hydrost. press}}$$

Model eqs

for the **vapour domain** $s(t) < x < L_s$ (same reasoning as before)

- $\frac{\partial}{\partial t}(\phi\rho_v) + \frac{\partial}{\partial x}(\phi\rho_v v_v) = 0, \qquad s(t) < x < L_s$
- ideal gas law for $\rho_v = \dfrac{P_v}{rT}$ implies

$$\frac{\partial}{\partial t}\left(\phi\frac{P_v}{rT}\right) + \frac{\partial}{\partial x}\left(\phi\frac{P_v}{rT}v_v\right) = 0 \qquad (16)$$

- Darcy's law $v_v = -\frac{K}{\phi\mu_v}\left(\frac{\partial P_v}{\partial x} + \rho_v g\right)$ implies

$$\frac{\partial}{\partial t}\left(\frac{P_v}{T}\right) - \frac{\partial}{\partial x}\left[\frac{P_v}{T}\frac{K}{\phi\mu_v}\left(\frac{\partial P_v}{\partial x} + \frac{P_v}{rT}g\right)\right] = 0,$$

- (assume viscosity μ_v independent of temperature T)

$$\frac{\partial P_v}{\partial t} - \frac{KT}{\phi\mu_v}\frac{\partial}{\partial x}\left[\frac{P_v}{T}\left(\frac{\partial P_v}{\partial x} + \frac{g}{r}\frac{P_v}{T}\right)\right] = 0.$$

Free boundary conditions

Problem: find P_l, P_v. We need two conditions at $x = s(t)$.

We impose **the continuity of mass flux** at the free boundary

$$[\rho_\beta (v_\beta - \dot{s})]_l^v = 0$$

use v_l, v_v found before to get

$$\dot{s}\left(1 - \frac{\rho_v}{\rho_l}\right) = \frac{K}{\phi \nu_v \rho_l}\left[\frac{PG}{rT}\left(\frac{\partial P_v}{\partial x} + \frac{P_v}{rT}g\right) - \frac{\nu_v}{\nu_l}\rho_l\left(\frac{\partial P_l}{\partial x} + \rho_l g\right)\right]$$

This gives the evolution of the liquid-vapour interface. We need a second condition.

Main question: *which pressure at the phase change interface* ?
It would be quite natural to assume

$$P_v|_{s(t)} = P_l|_{s(t)} = P^*|_{s(t)}\,, \qquad (17)$$

($P^*(T)$ =saturated vapour pressure). **However...**

Theorem. *Condition (17) is incompatible with the assumption of no-flux at the bottom boundary, unless* $\dot{s} = 0$ *identically.*
(Recall that we want study the long-time evolution!)

Alternative to (17): flux of linear momentum density preserved at $x = s(t)$

$$\chi v_v + P_v = \chi v_l + P_l$$

$\chi_\alpha := \rho_\alpha (v_\alpha - \dot{s})$ (net phase flux per unit mass crossing the interface)

Using the alternative

$$\chi\left(1-\frac{\rho_v}{\rho_l}\right)=\rho_v\left(v_v-v_l\right)$$

Reasonable assumption: $\rho_v \ll \rho_l$.

In this case $\chi \approx \rho_v v_v$ which in turn implies (**second interface condition**)

$$\rho_v v_v^2\big|_{s(t)} = P_l\big|_{s(t)} - P_v\big|_{s(t)}\,,$$

Notice: *difference* $P_l - P_v > 0$ *is very small (but not zero!) and can be estimated in terms of the mass flux.*

Theorem. $\dot{s}\cdot v_v < 0$ (i. e. $s(t)$ *moves downward if the vapour volumetric flux is upward and vice versa*)

Some parameter values

$L_s = -1300\,m$	$L_i = -3500\,m$
$T_s = 520\,^oK$	$T_i = 610\,^oK$
$s_{ip} \approx -3060\,m$	$T(s_{ip}) = 592\,^oK$
$\Delta L_v = L_s - s_{ip} \approx 1800\,m$	$\hat{T}(x \in [s_{ip}, L_s]) \approx 600^oK$
$\rho_{vc} = \frac{P^*(T(s_{ip}))}{rT} \approx 40Kg/m^3$	$\rho_l = 10^3 Kg/m^3$
$\phi = 10^{-2}$	$g = 9,8\,m/s$
$\mu_v \approx 2\times 10^{-5}\,Pa\cdot s$	$K = 10^{-16}m^2$
$r = 4,6\times 10^2 J/Kg^oK$	

Estimated Pressure:

$$P_v\left(x=s_{ip}\right) = P^*(T(x=s_{ip})) = P^*_{ip} \approx 1,1\times 10^7\,Pa$$

$$P_v\left(x=L_s\right) = P_s = 3,1\times 10^6\,Pa$$

$$\Delta P_v = P_v\left(x=L_s\right) - P_v\left(x=s_{ip}\right) \approx -8\times 10^6\,Pa$$

Estimated vapour velocity

Previous assumption implies

$$P_l|_{x=s(t)} = P_l(s(t)) = P^*|_{x=s(t)} + \rho_v v_v^2|_{x=s(t)} \, .$$

However using typical field data

$$|v_v| \approx \frac{K}{\phi \mu_v} \left| \frac{\Delta P}{L} \right| \approx 10^{-6} \, \frac{m}{s}$$

Thus

$$\rho_v v_v^2 \approx 10^{-13} \, Pa$$

We conclude that the pressure jump at the interface is very small and consequently $\dot{s} \approx 0$ (but analysis shows that if it is totally neglected then $s(t)$ remains steady).

The quasi-steady approximation

Continue to consider the simpler condition

$$P_l|_{x=s(t)} = P_v|_{x=s(t)} = P^*(T(x))|_{x=s(t)} \, .$$

i.e. pressure jump equal to zero, **but consider two different time scales** to account for the movement of the interface!

Scale analysis shows that: **the dynamics of vapour diffusion in $[s(t), L_s]$ occurs over a time scale for which the free boundary appears at rest. Vice versa over the time scale where the movement of the interface is not negligible diffusive effects appears at equilibrium.**

This suggests to use a **quasi stationary** approach to study the movement of $s(t)$.

Free boundary prob. with $P_v|_{x=s(t)} = P^*|_{x=s(t)}$,

$$\begin{cases} \dfrac{\partial P_v}{\partial t} - \dfrac{K\,T}{\phi\mu_v}\dfrac{\partial}{\partial x}\left[\dfrac{P_v}{T}\left(\dfrac{\partial P_v}{\partial x} + \dfrac{g}{r}\dfrac{P_v}{T}\right)\right] = 0, \\ P_v(x = L_s) = P_s, \\ P_v(x = s(t)) = P^*(s(t)), \\ \dot{s}\left(1 - \dfrac{P^*(s(t))}{rT\rho_l}\right) = \dfrac{P^*(s(t))}{rT\rho_l}\dfrac{K}{\phi\mu_v}\left(\dfrac{\partial P_v}{\partial x} + \dfrac{P^*(s(t))}{rT}g\right)\Big|_{x=s(t)}, \\ P_v(t = 0) = P_{in}(x), \\ s(t = 0) = s_{in}, \end{cases}$$

Here $T(x)$ is known, $P^*(x) = P^*(T(x))$

Possible simplifications for typical values

$$\beta_1(x) := \frac{\frac{P_v}{rT}g}{\frac{\partial P_v}{\partial x}} = \frac{\text{gravitational force}}{\text{pressure gradient}} \approx \frac{\frac{P_v}{rT}g}{\frac{\Delta P_v}{\Delta L_v}} \approx 10^{-1}$$

We cannot neglect the gravitational contribution!

Consider then the ratio *vapour density/liquid density* at the interface:

$$\beta_2 := \frac{\rho_v(x = s(t))}{\rho_l} = \frac{P^*(s(t))}{rT(s(t))\rho_l} \approx \frac{P^*_{ip}}{r\hat{T}\rho_l} \approx 10^{-2}$$

This justifies the assumption $\rho_v \ll \rho_l$ made before. We the neglect β_2 to get a **simplified free boundary equation**

$$\dot{s} = \frac{P^*(s(t))}{rT\rho_l}\frac{K}{\phi\mu_v}\left(\frac{\partial P_v}{\partial x} + \frac{P^*(s(t))}{rT}g\right)\Big|_{x=s(t)}.$$

Equations scaling

char. diff. time	**char. interf. time**	**for typical data**
$t_{\text{diff}} := \phi \mu_v L^2 / K P^*_{ip}$	$t_{\text{interf}} := \frac{\rho_l}{\rho_{vc}} t_{\text{diff}}$	$t_{\text{diff}} \approx 27$ years

(since $\rho_l / \rho_{vc} \approx 25$). Thus t_{interf} is of order **hundreds of years**.

Take as scale time *that of the interface*: **scaled (adimensionalized) equations**

$$\underbrace{\frac{t_{\text{diff}}}{t_{\text{interf}}}}_{\frac{\rho_{vc}}{\rho_l} \approx 4 \times 10^{-2}} \frac{\partial \tilde{P}_v}{\partial \tilde{t}} - \tilde{T} \frac{\partial}{\partial \tilde{x}} \left[\frac{\tilde{P}_v}{\tilde{T}} \left(\frac{\partial \tilde{P}_v}{\partial \tilde{x}} + \alpha \frac{\tilde{P}_v}{\tilde{T}} \right) \right] = 0,$$

$$\dot{\tilde{s}} = \frac{\tilde{P}^*(\tilde{x})}{\tilde{T}} \left(\frac{\partial \tilde{P}_v}{\partial \tilde{x}} + \alpha \frac{P^*(\tilde{x})}{T} \right) \Bigg|_{\tilde{x} = \tilde{s}(\tilde{t})} \qquad \left(\alpha := \frac{gL}{rT_i} \approx 10^{-1} \right)$$

Quasi-steady approximation at interface scale time

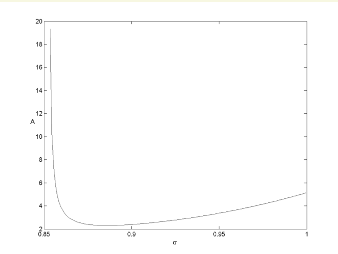
$$\begin{cases} \frac{\partial}{\partial \tilde{x}} \left[\frac{\tilde{P}_v}{\tilde{T}} \left(\frac{\partial \tilde{P}_v}{\partial \tilde{x}} + \alpha \frac{\tilde{P}_v}{\tilde{T}} \right) \right] = 0, \\ \tilde{P}_v(\tilde{x} = 0) = \tilde{P}_s, \\ \tilde{P}_v(\tilde{x} = \tilde{s}(\tilde{t})) = \tilde{P}^*(\tilde{s}(\tilde{t})), \\ \dot{\tilde{s}} = \frac{\tilde{P}^*(\tilde{x})}{\tilde{T}} \left(\frac{\partial \tilde{P}_v}{\partial \tilde{x}} + \alpha \frac{\tilde{P}^*(\tilde{x})}{T} \right) \Bigg|_{\tilde{x} = \tilde{s}(\tilde{t})}, \\ \tilde{P}_v(\tilde{t} = 0) = \tilde{P}_{in}(\tilde{x}), \\ \tilde{s}(\tilde{t} = 0) = \tilde{s}_0. \end{cases} \tag{18}$$

Some implications

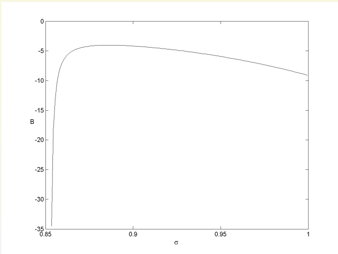
Being T a known linear function of x the quasi-steady problem can be written as a function of T and easily integrated:

$$P = \sqrt{\frac{A}{1-\delta}T^2 + BT^{2\delta}}, \qquad \delta = \alpha/(1-T_s)$$

$T_s = 0.85$ is the temperature at basin top boundary, A and B are known functions of $\sigma(t) := T(s(t))$.



(a) A as a function of σ



(b) B as a function of σ

Both functions diverges as $\sigma \to T_s$.

Reparametrized free boundary eq

$$\dot{\sigma} = A(\sigma)\gamma^2, \qquad \sigma(0) = \sigma_0,$$

If $s(t)$ is sufficiently far from the the basin top boundary then $A(\sigma) \approx A_0 - A(s_0)$ and

$$\sigma(t) = A_0\gamma^2 t + \sigma_0,$$

This allows **a better estimate** of the characteristic interface time

$$t_{\text{interf,new}} := \frac{t_s}{A_0\gamma^2} \approx \frac{\rho_l}{\rho_{vc}}\frac{1}{A_0\gamma^2}\,t_{\text{diff}}$$

Being $A_0 \approx 4$ and $\gamma^2 \approx 2,2 \times 10^{-2}$ we get

$$t_{\text{interf,new}} \approx 2,8 \times 10^2\, t_{\text{diff}} \approx \textbf{7500 years}$$

Some conclusions

This seems to be the time needed for the Larderello basin to evolve from water-dominated to vapour-dominated.

This appears **compatible with geological studies**: Larderello is an “old” basin. Thousands of years ago it was a water dominated basin which has now turned to a vapour-dominated one.

Numerical simulations carried out with the complete quasi-steady problem confirm these long-time evolution also for Monteversi Marittimo (still in the transient mixed state).

Essential bibliography

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Thanks for your attention!